Hecke Theory and Jacquet Langlands

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Today we're going to be associating L-functions to automorphic things and discussing their L-function-y properties, i.e. analytic continuation and functional equation. The point is that L-functions will help us understand automorphic things, e.g. we can compare automorphic things by comparing their L-functions, and we can use properties of L-functions to detect whether our automorphic thing is cuspidal. We follow Chapter 6 of Gelbart's book *Automorphic forms on adele groups*.

1 GL_1 by Hecke and Tate

For motivation/illustration we'll start with the case of GL_1, which is the work of Hecke, as well as Tate's thesis.

Let F be a number field, and $\mathbb{A}_F$ its ring of adeles. An automorphic representation of GL_1 (over F) is any irreducible unitary representation of $\mathrm{GL}_1(\mathbb{A}_F)$ appearing in $L^2(\mathrm{GL}_1(F)\backslash \mathrm{GL}_1(\mathbb{A}_F))$. But as a $\mathrm{GL}_1(\mathbb{A}_F)$-module,

$$L^2(\mathrm{GL}_1(F)\backslash \mathrm{GL}_1(\mathbb{A}_F)) = \int^{\oplus}_{\mathrm{GL}_1(F)\backslash \mathrm{GL}_1(\mathbb{A}_F)} \psi(g)$$

so an automorphic representation is simply a grossencharacter of F, i.e. a character $\psi : F^\times \backslash \mathbb{A}_F^\times \to S^1$. Thus the theory we're looking for is, in the case of GL_1, simply Hecke's theory of L-functions associated to grossencharacters. We review this now.

Let $\psi : \mathbb{A}_F^\times \to S^1$ (trivial on $F^\times$) be a grossencharacter of F. We can decompose it

$$\psi = \prod_v \psi_v$$

as a product of characters ψ_v on $F_v^\times$, with ψ_v unramified for almost all v.

Let S be the set of ramified places. For every $v \notin S$, define

$$\chi(v) = \psi_v(\pi_v)$$

where $\pi_v \in F_v$ is a uniformizer (note that the choice of uniformizer does not matter). We can extend χ by multiplicativity to the set of ideals of F prime to S.

The L-series associated to ψ, or to χ, is

$$L(s,\chi) = \sum_{S \nmid \mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{v \notin S} \left(1 - \frac{\chi(v)}{N(v)^s}\right)^{-1}.$$

Theorem (Hecke)**.** *$L(s,\chi)$ has nice L-function-y properties. That is:*

1. *it converges for $\Re(s) > 1$ (easy);*
2. *it has a meromorphic continuation to the whole plane, with a simple pole at $s = 1$ if χ is trivial and no poles otherwise;*
3. *there is a constant A, a constant $W(\chi)$ of modulus 1, and a gamma factor $\Gamma(s,\chi)$ such that $R(s,\chi) = s(s-1)A^s\Gamma(s,\chi)L(s,\chi)$ is entire and satisfies the functional equation*

$$R(1-s,\chi^{-1}) = W(\chi)R(s,\chi).$$

Now Tate's thesis. Let $S(\mathbb{A}_F)$ be the space of *Schwartz-Bruhat functions* on $\mathbb{A}_F$, that is, functions of the form $f = \prod_v f_v$ where

- f_v is a Schwartz function (in the usual sense of rapidly decreasing) on $\mathbb{R}$ or $\mathbb{C}$ if v is real or complex,
- f_v is locally constant and compactly supported on F_v if v is finite, and
- $f_v = \mathbf{1}_{\mathcal{O}_v}$ for almost all v.

For a non-trivial character τ of $\mathbb{A}_F$, define a Fourier transform on $S(\mathbb{A}_F)$ by

$$\hat{f}(\alpha) = \int_{\mathbb{A}_F} f(\beta)\tau(\alpha\beta)d\beta.$$

For ψ a grossencharacter of F, define a global ζ-function by

$$\zeta(f,\psi,s) = \int_{\mathrm{GL}_1\,\mathbb{A}_F} f(\alpha)\psi(\alpha)|\alpha|^s d^\times\alpha$$

where $|\alpha| = \prod_v |\alpha_v|_v$ and $d^\times\alpha = \prod_v d^\times\alpha_v$.

Since everything involved is a product of local factors, we can decompose our global ζ-function

$$\zeta(f,\psi,s) = \prod_v \zeta(f_v,\psi_v,s)$$

into local ζ-functions

$$\zeta(f_v,\psi_v,s) = \int_{\mathrm{GL}_1\,F_v} f_v(\alpha_v)\psi_v(\alpha_v)|\alpha_v|_v^s d^\times\alpha_v$$

(as long everything converges, e.g. for $\Re(s) > 1$).

Theorem (Tate)**.** *$\zeta(f,\psi,s)$ has a meromorphic continuation to the whole complex plane, with simple poles at $s = 0$ and $s = 1$ if ψ is trivial and entire if not; furthermore it satisfies the functional equation*

$$\zeta(f,\psi,s) = \zeta(\hat{f},\psi^{-1},1-s).$$

This is proven almost entirely using harmonic analysis (i.e. Fourier analysis) on $M(1,F)$ and GL_1.

The theorem of Tate implies the theorem of Hecke via the relation

$$\prod_{v\notin S_\psi} \zeta(f_v,\psi_v,s) = L(s,\chi)\prod_{v\notin S_\psi} \chi(\partial_v)^{-1}N(\partial_v)^{s-1/2}$$

for the correct f_v, where ∂_v is the different of F_v, and as above S_ψ is the set of primes at which ψ ramifies and χ is the Hecke character associated to ψ. That is, this relation can be used to prove the analytic continuation and functional equation of $L(s,\chi)$ from the corresponding properties of $\zeta(f_v,\psi_v,s)$.

2 GL_2 by Jacquet and Langlands

Just as for GL_1, we now want to associate L-functions to automorphic forms for GL_2. We already know how to do this for classical holomorphic cusp forms: if $f \in S_k(N, \psi)$ has Fourier expansion $f = \sum a_n q^n$ at infinity, then we can define an L-function by

$$L(f,s) = \frac{1}{(2\pi)^s}\Gamma(s) \sum_{n\geq 1} \frac{a_n}{n^s} = \int_0^\infty f(iy) y^{s-1} dy,$$

that is, by taking the Mellin transform of f along the vertical half-line $\{iy : y > 0\}$. To see how to define our L-functions more generally we rewrite this in the adelic setting.

Let

$$\phi_f(g) = f(g_\infty(i)) j(g_\infty, i)^{-k} \psi(k_0), \qquad g = \gamma g_\infty k_0 \in \mathrm{GL}_2 \mathbb{A} = G_{\mathbb{Q}} G_\infty^+ K_0$$

be the adelic automorphic form associated to f. For simplicity suppose $N = 1$ and ψ is trivial. Then $\phi_f(g)$ is right K_0-invariant, and (as always) left $\mathrm{GL}_2 \mathbb{Q}$-invariant. From the definition of ϕ_f we see, for real $y > 0$,

$$\phi_f\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = f(iy).$$

Thus

$$L(f,s) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi_f\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) |y|^s d^\times y.$$

Now we get Fourier analysis involved. Recall that the characters of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ are given by $\tau(\lambda x)$ for various $\lambda \in \mathbb{Q}^\times$, where

$$\tau(x) = \prod_{p \leq \infty} \tau_p(x)$$

and $\tau_\infty(x) = e^{2\pi i x_\infty}$ and $\tau_p(x) = 1$ if and only if $x_p \in \mathcal{O}_p$. Thus $\phi_f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)$ has a Fourier expansion as a function of x,

$$\phi_f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \sum_{\lambda \in \mathbb{Q}} \phi_{f,\lambda}(g) \tau(\lambda x),$$

where

$$\phi_{f,\lambda}(g) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi_f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) \overline{\tau(\lambda x)} dx$$

is the λ-th Fourier coefficient of $\phi_f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g)$ (depending on g).

Lemma. *Suppose $f \in S_k(\Gamma(1))$ has Fourier expansion $f = \sum a_n e^{2\pi i n z}$, and ϕ_f is the associated adelic automorphic form. Then for real $y > 0$,*

$$\phi_{f,\lambda}(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) = \begin{cases} a_n e^{-2\pi n y} & \text{if } \lambda = n \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

Letting $x = 0$, so that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \mathrm{id}$ and $\tau(\lambda x) = 1$, and defining

$$W_{\phi_f}(g) = \int_{\mathbb{Q} \backslash \mathbb{A}} \phi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) \overline{\tau(x)} dx$$

to be the 1st Fourier coefficient of ϕ_f, we find

$$\phi_f(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) = \sum_{\lambda \in \mathbb{Q}^\times} \phi_{f,\lambda}(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) = \sum_{\lambda \in \mathbb{Q}^\times} W_{\phi_f}(\begin{pmatrix} \lambda y & 0 \\ 0 & 1 \end{pmatrix}).$$

Now our L-function becomes

$$L(f,s) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \sum_{\lambda \in \mathbb{Q}^\times} W_{\phi_f}(\begin{pmatrix} \lambda y & 0 \\ 0 & 1 \end{pmatrix}) |y|^s d^* y = \int_{\mathbb{A}^\times} W_{\phi_f}(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}) |y|^s d^* y.$$

In words, $L(f,s)$ is the adelic Mellin transform along $\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ of the 1st Fourier coefficient of ϕ_f. This is supposed to suggest that in general, the L-function associated to an automorphic representation π should be the Mellin transform of the first Fourier coefficient of some distinguished function in the space of π.

3 Whittaker Models and L-functions

In order to find what such a function should be, let's list some properties of the one W_{ϕ_f} in the case of a classical cusp form f. The right-translates (or more precisely we may want right convolutions) of $W_{\phi_f}(g)$ generate a space $W(\pi_f)$ of functions W on $\mathrm{GL}_2 \mathbb{A}$ satisfying:

- $W(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \tau(x) W(g)$ for $x \in \mathbb{A}$;
- $W(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} g) = \psi(z) W(g)$ for $z \in \mathbb{A}^\times$;
- W is right K-finite, C^∞ as a function on G_∞, and rapidly decreasing;
- the representation of $\mathscr{H}(\mathrm{GL}_2 \mathbb{A})$ on $W(\pi_f)$ given by right convolution is equivalent to π_f.

The (equivalent) representation $W(\pi_f)$ is called the *Whittaker model* of π_f, and the space $W(\pi_f)$ the *Whittaker space*.

Now to produce an L-function we can produce a distinguished function in $W(\pi_f)$ and then Mellin transform it. In fact we will produce a distinguished function locally, on the Whittaker models of local factors of our automorphic representation, and then glue them together to get a distinguished function in the global Whittaker model. Note that the Whittaker model lends itself to this task, because it is composed of functions on $\mathrm{GL}_2 \mathbb{A}$, rather than functions on $\mathrm{GL}_2 \mathbb{Q} \backslash \mathrm{GL}_2 \mathbb{A}$, so the local factors fit together nicely.

We now discuss Whittaker models more generally.

Theorem. *Let v be any finite place of F, and π_v an irreducible admissible (infinite-dimensional) representation of $\mathrm{GL}_2 F_v$. Then in the space of locally constant functions W on $\mathrm{GL}_2 F_v$ satisfying*

$$W(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \tau(x) W(g) \qquad x \in F_v, g \in \mathrm{GL}_2 F_v$$

there is a unique subspace $W(\pi_v)$ stable under the right action of $\mathrm{GL}_2 F_v$ and equivalent as a $\mathrm{GL}_2 F_v$-module to π_v.

If v is archimedean, there is a unique subspace of the space of C^∞ functions W satisfying the above condition and also

$$W(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = O(|t|)^N \textit{ as } |t| \to \infty$$

equivalent to π_v as an $\mathscr{H}(\mathrm{GL}_2 F_v)$-module.

As before $W(\pi_v)$ is called the *Whittaker space* of π_v, and the representation on $W(\pi_v)$ is called the *Whittaker model*.

Global Whittaker models are analogous. Local uniqueness implies global uniqueness, and this implies multiplicity one for automorphic representations. Indeed, the association $\phi \mapsto W_\phi$ produces a Whittaker model from an automorphic representation, and the uniqueness of the Whittaker model implies uniqueness of the automorphic representation.

To every function in the local Whittaker model we associate a ζ-function, a special one of which will be our local L-function. Let π_v be an irreducible admissible representation of $\mathrm{GL}_2 F_v$, χ a unitary character of $F_v^\times$, $g \in \mathrm{GL}_2 f_v$, $W \in W(\pi_v)$. Then define a local ζ-function for all of this data by

$$\zeta(g, \chi, W, s) = \int_{F_v^\times} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})\chi(a)|a|^{s-1/2} d^\times a.$$

Theorem 1.

- *The integral defining $\zeta(g, \chi, W, s)$ converges in some right half-plane.*
- *There is a $W^0 \in W(\pi_v)$ such that $L(\chi \otimes \pi_v, s) = \zeta(1, \chi, W^0, s)$ is an Euler factor making $\frac{\zeta(g,\chi,W,s)}{L(\chi\otimes\pi_v,s)}$ entire for every g, χ, W.*
- *$\zeta(g, \chi, W, s)$ has an analytic continuation to the whole plane satisfying the functional equation*

$$\frac{\zeta(g, \chi, W, s)}{L(\chi \otimes \pi_v, s)} \varepsilon(\pi_v, \chi, s) = \frac{\zeta(wg, \chi^{-1}\psi_v^{-1}, W, 1-s)}{L(\chi^{-1}\psi_v^{-1} \otimes \pi_v, 1-s)}$$

for some function $\varepsilon(\pi_v, \chi, s)$ independent of g, W, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, ψ_v is the central character of π_v.

For a finite place v, "Euler factor" means $\frac{1}{P(q^s)}$ where P is a polynomial with constant term 1 and $q = |\omega_v|$; for an infinite place, it means some kind of Γ-factor. The ε factor is also not too bad, just an exponential type thing.

The L-function associated to an automorhic representation π will be the product of the local L-factors associated to its local factors π_v; it will also be the Mellin transform of the distinguished function given by the product of the local parts W^0.